

An Approach to Maxwell Equations in Uniformly Accelerated Spherical Coordinates by Newman-Penrose Method

Z. Ya. Turakulov

Institute of Nuclear Physics

Ulugbek, Tashkent 702132, Rep. of Uzbekistan, CIS

(e-mail:zafar@.suninp.tashkent.su)

Variables are separated in Maxwell equations by the Newman-Penrose method of isotropic complex tetrad in the uniformly accelerated spherical coordinate system. Particular solutions are obtained in terms of spin 1 spherical harmonics. PACS: 03.50.De

1 Introduction

The uniformly accelerated spherical coordinates have been introduced in our work [1]. It turns out that the Newman-Penrose method of isotropic complex tetrad [2] works in this coordinate system. The aim of the present work is to demonstrate applying the method in the coordinate system and obtain the general solution to Maxwell equations in these coordinates.

Although a comprehensive formulation of the method have been published [3], in order to apply it in a new coordinate system we have to start with composing the tetrad and thus to repeat the whole exposition. Thus, the method is to be actually recovered in details.

2 The isotropic complex tetrad

The metric for the uniformly accelerated spherical coordinates $\{\xi, u, v, \varphi\}$ has the form [1]

$$ds^2 = a^2 \frac{\sinh^2 u d\xi^2 - du^2 - dv^2 - \sin^2 v d\varphi^2}{(\cosh u + \cos v)^2}. \quad (1)$$

To compose the isotropic complex tetrad we first solve the Hamilton-Jacobi equation for isotropic lines in this metric:

$$(\cosh u + \cos v)^2 \left[\frac{1}{\sinh^2 u} \left(\frac{\partial S}{\partial \xi} \right)^2 - \left(\frac{\partial S}{\partial u} \right)^2 - \left(\frac{\partial S}{\partial v} \right)^2 + \frac{1}{\sin^2 v} \left(\frac{\partial S}{\partial \varphi} \right)^2 \right] = 0.$$

Substituting $S = E\xi + U(u) + V(v) + M\varphi$ yields a separated equation:

$$E^2 \sinh^{-2} u - U'^2 = V'^2 - M^2 \sin^{-2} v = L^2.$$

The two congruences of isotropic rays to be used below for composing the tetrad corresponds to the values $E = 1, L = M = 0$ and $M = 1, L = E = 0$ of the constants. They give respectively

$$U' = \pm \sinh^{-1} u, \quad V' = 0$$

and

$$U' = 0, \quad V' = \pm \sin^{-1} v.$$

The isotropic co-vectors with these two congruences are normalized with respect to the metric

$$ds^2 = a^2 (\sinh^2 u d\xi^2 - du^2 - dv^2 - \sin^2 v d\varphi^2)$$

obtained from the metric (1) by an apparent conformal transformation. It is convenient to do so because Maxwell equations are forminvariant under

conformal transformations. After the normalization procedure the tetrad appears in the form:

$$\kappa = \sinh u d\xi - du, \quad \lambda = \sinh u d\xi + du \quad (2)$$

$$\mu = dv + \imath \sin v d\varphi, \quad \nu = dv - \imath \sin v d\varphi$$

The co-vectors constitute a normalized orthogonal frame with only two non-zero scalar products: $\langle \kappa, \lambda \rangle = \langle \mu, \nu \rangle = 1$. The reciprocal relations are:

$$d\xi = \frac{\kappa + \lambda}{2 \sinh u}, \quad du = \frac{\lambda - \kappa}{2} \quad (3)$$

$$dv = \frac{\mu + \nu}{2}, \quad d\varphi = \imath \frac{\mu - \nu}{2 \sin v}$$

The vector frame dual to the co-vector frame (1) is the following:

$$\vec{k} = \frac{1}{2} \left(\frac{1}{\sinh u} \frac{\partial}{\partial \xi} - \frac{\partial}{\partial u} \right), \quad \vec{l} = \frac{1}{2} \left(\frac{1}{\sinh u} \frac{\partial}{\partial \xi} + \frac{\partial}{\partial u} \right), \quad (4)$$

$$\vec{m} = \frac{1}{2} \left(\frac{\partial}{\partial v} - \frac{\imath}{\sin v} \frac{\partial}{\partial \varphi} \right), \quad \vec{n} = \frac{1}{2} \left(\frac{\partial}{\partial v} + \frac{\imath}{\sin v} \frac{\partial}{\partial \varphi} \right)$$

Consider the following 2-forms Φ^1, Φ^2, Φ^3 :

$$\Phi^1 = \kappa \wedge \mu, \quad \Phi^2 = \frac{1}{2}(\kappa \wedge \lambda + \mu \wedge \nu), \quad \Phi^3 = \lambda \wedge \nu \quad (5)$$

Evaluating them due to the relations (2) gives:

$$\Phi^1 = \sinh u d\xi \wedge dv + \imath \sinh u \sin v d\xi \wedge d\varphi - du \wedge dv + \imath \sin v d\varphi \wedge du \quad (6)$$

$$\Phi^2 = \sinh u d\xi \wedge du - \imath \sin v dv \wedge d\varphi$$

$$\Phi^3 = \sinh u d\xi \wedge dv - \imath \sinh u \sin v d\xi \wedge d\varphi + du \wedge dv + \imath \sin v d\varphi \wedge du$$

It is seen that the frame of 2-forms $\{\Phi^a\}$ is self-dual:

$$*\Phi^a = \imath \Phi^a. \quad (7)$$

As these three 2-forms are linearly independent they constitute a complete frame.

Exterior derivatives of Φ^a 's evaluated from the equations (5) with inserting the expressions (3) and rewritten in terms of the tetrad (1) and the frame $\{\Phi^a\}$ are:

$$d\Phi^1 = \frac{1}{2} \coth u \lambda \wedge \Phi^1 + \cot v \nu \wedge \Phi^1, \quad d\Phi^2 = 0 \quad (8)$$

$$d\Phi^3 = -\frac{1}{2} \coth u \kappa \wedge \Phi^3 + \cot v \mu \wedge \Phi^3.$$

3 Reduction of Maxwell equations

An arbitrary 2-form of strengths of electromagnetic field E can be represented as a expansion in the frame of Φ^a 's:

$$E = F\Phi^1 + G\Phi^2 + H\Phi^3 \quad (9)$$

with F , G and H being arbitrary scalar functions. Since, due to the equations (7) the 2-form E is self-dual Maxwell equations are reduced to one equation $dE = 0$:

$$0 = dE = (\vec{l} \circ F)\lambda \wedge \Phi^1 + (\vec{n} \circ F)\nu \wedge \Phi^1 + (\vec{k} \circ H)\kappa \wedge \Phi^3 + (\vec{m} \circ H)\mu \wedge \Phi^3 + \quad (10)$$

$$(\vec{k} \circ G)\kappa \wedge \Phi^2 + (\vec{l} \circ G)\lambda \wedge \Phi^2 + (\vec{m} \circ G)\mu \wedge \Phi^2 + (\vec{n} \circ G)\nu \wedge \Phi^2 + Fd\Phi^1 + Hd\Phi^3$$

where action of vectors on scalars is the same as that of differential operators (4). It is convenient to employ ambiguity of expression of 3-forms as exterior products of the tetrad elements and Φ^a 's to eliminate the 2-form Φ^2 :

$$\begin{aligned} \kappa \wedge \Phi^2 &= \frac{1}{2}\nu \wedge \Phi^1, \quad \lambda \wedge \Phi^2 = -\frac{1}{2}\mu \wedge \Phi^3, \\ \mu \wedge \Phi^2 &= -\frac{1}{2}\lambda \wedge \Phi^1, \quad \nu \wedge \Phi^2 = \frac{1}{2}\kappa \wedge \Phi^3. \end{aligned}$$

After eliminating 3-forms containing Φ^2 , collecting similar terms, inserting the expressions (8) and annullating the common factors one can rewrite the equation (10) in the form

$$\begin{aligned} \left(\frac{\partial}{\partial u} + \frac{1}{\sinh u} \frac{\partial}{\partial \xi} + \coth u \right) F &= \frac{1}{2} \left(\frac{\partial}{\partial v} - \frac{\imath}{\sin v} \frac{\partial}{\partial \varphi} \right) G \\ \left(\frac{\partial}{\partial v} + \frac{\imath}{\sin v} \frac{\partial}{\partial \varphi} + \cot v \right) F &= -\frac{1}{2} \left(\frac{\partial}{\partial u} - \frac{1}{\sinh u} \frac{\partial}{\partial \xi} \right) G \\ \left(\frac{\partial}{\partial u} - \frac{1}{\sinh u} \frac{\partial}{\partial \xi} + \coth u \right) H &= -\frac{1}{2} \left(\frac{\partial}{\partial v} + \frac{\imath}{\sin v} \frac{\partial}{\partial \varphi} \right) G \\ \left(\frac{\partial}{\partial v} - \frac{\imath}{\sin v} \frac{\partial}{\partial \varphi} + \cot v \right) H &= \frac{1}{2} \left(\frac{\partial}{\partial u} + \frac{1}{\sinh u} \frac{\partial}{\partial \xi} \right) G \end{aligned}$$

To accomplish the further reduction we put:

$$F = f_+ e^{\imath(k\xi + m\varphi)} + f_- e^{-\imath(k\xi + m\varphi)} \quad (11)$$

$$\frac{1}{2}G = g e^{\imath(k\xi + m\varphi)} + g e^{-\imath(k\xi + m\varphi)}$$

$$H = f_- e^{\imath(k\xi + m\varphi)} + f_+ e^{-\imath(k\xi + m\varphi)}$$

and have two coinciding pairs of equations which are

$$\left(\frac{\partial}{\partial u} + \coth u \pm \frac{\imath k}{\sinh u} \right) f_{\pm} = \left(\frac{\partial}{\partial v} \pm \frac{m}{\sin v} \right) g \quad (12)$$

$$\left(\frac{\partial}{\partial v} + \cot v \pm \frac{m}{\sin v} \right) f_{\pm} = \left(\frac{\partial}{\partial u} \mp \frac{\imath k}{\sinh u} \right) g$$

This system can be solved first for the functions f and then the function g can be found from these equations.

4 Variables separation and explicit form of the scalar functions

The equations (11) reduce to the following equation for the functions f_{\pm} :

$$\left[\left(\frac{\partial^2}{\partial u^2} + \coth u \frac{\partial}{\partial u} + \frac{k^2 \mp \imath k \cosh u + 1}{\sinh^2 u} \right) + \left(\frac{\partial^2}{\partial v^2} + \cot v \frac{\partial}{\partial v} + \frac{m^2 \pm m \cos v - 1}{\sin^2 v} \right) \right] f_{\pm} = 0.$$

Taking the function to be found in factorized form

$$f_{\pm} = U_{\pm}(u) V_{\pm}(v) \quad (13)$$

Substituting this separates the equation and gives:

$$U_{\pm}'' + U_{\pm}' \coth u + \frac{k^2 \pm \imath k \cosh u + 1}{\sinh^2 u} = l(l+1) U_{\pm}$$

$$V_{\pm}'' + V_{\pm}' \cot v - \frac{m^2 \pm m \cos v + 1}{\sin^2 v} = -l(l+1) V_{\pm}.$$

Solutions of these equations are known as spin 1 spherical harmonics [3]:

$$U_{\pm}(u) = {}_1S_{l \ \imath k}(u) \quad V_{\pm}(v) = {}_1S_{lm}(v).$$

Due to the equations (11) the function g is

$$g(u, v) = \frac{1}{l^2(l+1)^2} \left(\frac{\partial}{\partial u} + \coth u \pm \frac{\imath k}{\sinh u} \right) {}_1S_{l \ \imath k}(u) \left(\frac{\partial}{\partial v} + \cot v \pm \frac{m}{\sin v} \right) {}_1S_{lm}(v).$$

Substituting this into the equations (13) and, further, into equations (11) together with the equations (9) and (6) one obtains particular solutions of Maxwell equations, forming a complete orthogonal basis in the functional space. In the case $m = 0$ one obtains the expansion found in our work [1].

References

- [1] Z Y Turakulov, *Turkish J of Phys* 18 (1994) 479; Z Y Turakulov, *Geometry and Physics* 14 (1994)305
- [2] E T Newman, R Penrose. *J Math Phys* 14 (1973) 874
- [3] J N Goldberg, A J Macfarlane, E T Newman, F Rohrlich, E C G Sudarshan. *J Math Phys* 8 (1967) 2155